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# B. Sc. Part II, Paper 3

## Infinite Series

1. Comparison test - If $\Sigma u_n$ and $\Sigma v_n$ be two series of positive terms, then

$\underset{n\to\infty}{Lt} \frac{u_n}{v_n}$ be finite and non-zero.

The series will be both convergent or both divergent.

2. The infinite series

$\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$ to $\infty$ is convergent if $p>1$ and divergent if $p\le 1$.

Working process

(i) Find the $n^{th}$ term $u_n$.

(ii) Consider another auxiliary series whose $n^{th}$ term $v_n$ is equal to

$$v_n = \frac{\text{term of highest power of } n \text{ in numerator of } u_n}{\text{term of highest power of } n \text{ in the denominator of } u_n}$$

(iii) Find $\frac{u_n}{v_n}$

(iv) Find $\underset{n\to\infty}{Lt} \frac{u_n}{v_n}$

If $\underset{n\to\infty}{Lt} \frac{u_n}{v_n}$ = finite and non-zero

then proceed as follow.

(v) Compare $v_n$ with the series $\Sigma \frac{1}{n^p}$.

$\Sigma \frac{1}{n^p}$ is convergent if $p>1$.

is divergent if $p\le 1$.

(vi) By comparison test both the series $\Sigma U_n$ and $\Sigma V_n$ will converge or diverge simultaneously.

<u>Example 1.</u> Test the convergency of the series

$$1+\frac{1+2}{1+2^2}+\frac{1+3}{1+3^2}+\cdots+\frac{1+n}{1+n^2}+\cdots \text{ to } \infty$$

<u>Soln</u>. We have

$$1+\frac{1+2}{1+2^2}+\frac{1+3}{1+3^2}+\cdots+\frac{1+n}{1+n^2}+\cdots \text{ to } \infty$$

$$=\frac{1+1}{1+1^2}+\frac{1+2}{1+2^2}+\frac{1+3}{1+3^2}+\cdots+\frac{1+n}{1+n^2}+\cdots \text{ to } \infty$$

Then its $n^{th}$ term $= U_n = \frac{1+n}{1+n^2}$

Let us consider another series whose $n^{th}$ term is $V_n$ and $V_n = \frac{n}{n^2} = \frac{1}{n}$

$$\therefore \frac{U_n}{V_n} = \frac{\frac{1+n}{1+n^2}}{\frac{1}{n}} = \frac{n(1+n)}{1+n^2} = \frac{n+n^2}{1+n^2} = \frac{n^2\left(\frac{1}{n}+1\right)}{n^2\left(\frac{1}{n^2}+1\right)}$$

$$= \frac{1+\frac{1}{n}}{1+\frac{1}{n^2}}$$

$$\therefore \lim_{n\to\infty} \frac{U_n}{V_n} = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{1+\frac{1}{n^2}} = 1$$

which is finite and non-zero.

So, by comparison test, the two series $\Sigma U_n$ and $\Sigma V_n$ behave alike (converge or diverge simultaneously)

Now, $V_n = \frac{1}{n} \Rightarrow \Sigma V_n = \Sigma \frac{1}{n}$

$\Rightarrow \Sigma V_n = \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}+\cdots$ to $\infty$

Comparing it with $\Sigma \frac{1}{n^p}$, we have $p=1$

$\Rightarrow \Sigma V_n$ is not convergent i.e. divergent

Hence by comparison test $\Sigma U_n$ is divergent.